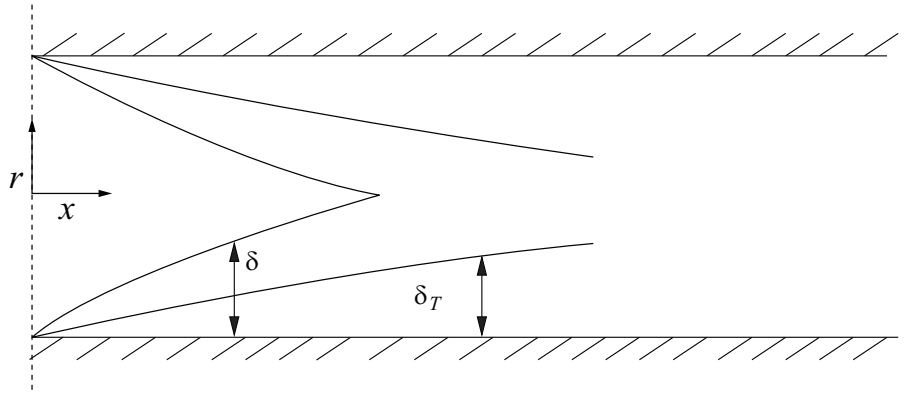

Figure 3.6 Hydrodynamic and thermal boundary layer thickness for $Pr >> 1$

Inverse is true for $Pr < 1$, $Pr^{1/2} \sim (\delta/\delta_T)$ and consequently δ_T is greater than δ. However, for extremely high Prandtl number fluids (such as oil), $Pr \geq 100$, $x_{e,h} << x_{e,t}$. Throughout the thermal entrance region, hydrodynamically fully developed velocity profiles can be assumed (Fig. 3.6).

For turbulent flow, conditions are nearly independent of Prandtl number. We can accept $\left(\frac{x_{e,t}}{D}\right)_{tur} \approx 10$. Bulk mean temperature at any section (x) is given by

$$T_m = \frac{2}{u_m r_0^2} \int_0^{r_0} u(r)\ T(r)\ r dr \tag{3.21}$$

3.3.1 Newton's Law of Cooling for Internal Flows

$$q_w'' = h(T_w - T_m) \tag{3.22}$$

The bulk mean temperature T_m is a convenient reference temperature for internal flows. T_m behaves like T_∞ for external flows. Essential difference is T_m varies in the flow direction. T_m increases in the flow direction if heat transfer is from surface to fluid. It is very important to know the variation of T_m under various hydrodynamic and thermal conditions, since the energy transfer is dependent on it.

3.3.2 Fully Developed Thermal Conditions

Heat transfer between the surface and the fluid results in a condition that the fluid temperature must continue to change with x. One might wonder about existence fully developed thermal conditions under such a condition. The situation is different

from the hydrodynamic case, for which $(\partial u/\partial x) = 0$ for the fully developed flow. In the presence of heat transfer, (dT_m/dx) as well as $(\partial T/\partial x)$ at any radial location are not zero.

We are actually introducing a new variable $(T_w - T)/(T_w - T_m)$, the condition for which this ratio becomes independent of x is of paramount interest. Although the temperature profile $T(r)$ continues to change with x, the relative shape of the profile does not change and the flow is said to be fully developed. Instead of $(dT_m/dx) = 0$ or $(dT/dx) = 0$, the condition is

$$\frac{d}{dx}\left[\frac{T_w(x) - T(r,x)}{T_w(x) - T_m(x)}\right] = 0 \tag{3.23}$$

The above equation signifies that $(T_w - T)$ changes in the same way as $(T_w - T_m)$ evolves.

Now, we can write

$$\frac{(T_w - T_m)\left(\frac{dT_w}{dx} - \frac{dT}{dx}\right) - (T_w - T)\left(\frac{dT_w}{dx} - \frac{dT_m}{dx}\right)}{(T_w - T_m)^2} = 0$$

or

$$\frac{1}{(T_w - T_m)}\frac{dT_w}{dx} - \frac{dT}{dx}\frac{1}{(T_w - T_m)} - \frac{T_w - T}{(T_w - T_m)^2}\frac{dT_w}{dx} + \frac{(T_w - T)}{(T_w - T_m)^2}\frac{dT_m}{dx} = 0$$

or

$$\frac{dT_w}{dx} - \frac{dT}{dx} - \frac{T_w - T}{T_w - T_m}\frac{dT_w}{dx} + \frac{(T_w - T)}{(T_w - T_m)}\frac{dT_m}{dx} = 0$$

or

$$\frac{dT}{dx} = \frac{dT_w}{dx} - \frac{T_w - T}{T_w - T_m}\frac{dT_w}{dx} + \frac{(T_w - T)}{(T_w - T_m)}\frac{dT_m}{dx} \tag{3.24}$$

Two surface conditions are available: uniform heat flux ($q_w'' = $ constant), and uniform surface temperature, $(dT_w/dx = 0)$. To achieve uniform heat flux, the tube wall should be heated electrically or the outer surface should be uniformly irradiated. If a phase change were taking place at the outer surface, the condition becomes $dT_w/dx = 0$ (uniform surface temperature).

For uniform wall heat flux, we get

$$\frac{dT_w}{dx} = \frac{dT_m}{dx} \tag{3.25}$$

Substituting this condition in the above equation for dT/dx, we get

$$\left.\frac{dT}{dx}\right|_{\text{fully developed}} = \left.\frac{dT_m}{dx}\right|_{\text{fully developed}} \tag{3.26}$$

For uniform wall temperature, $\frac{dT_w}{dx} = 0$

$$\left.\frac{dT}{dx}\right|_{fd,t} = \frac{(T_w - T)}{(T_w - T_m)} \left.\frac{dT_m}{dx}\right|_{fd,t} \tag{3.27}$$

Some important features of thermally developed flow may be extracted from Eq. (3.23). Since nondimensional temperature profile is independent of x, derivative of the profile with respect to r (temperature gradient from wall to fluid) must also be independent of x. Evaluating this derivative at the tube surface, we obtain

$$\frac{\partial}{\partial r}\left[\frac{T_w - T}{T_w - T_m}\right]_{r=r_0} = \frac{-\partial T/\partial r|_{r=r_0}}{(T_w - T_m)} \neq f(x) \tag{3.28}$$

Again we know,

$$q_w'' = k\frac{\partial T}{\partial r}|_{r=r_0} = h(T_w - T_m)$$

or

$$\frac{\frac{\partial T}{\partial r}|_{r=r_0}}{(T_w - T_m)} = \frac{h}{k} \tag{3.29}$$

From Eqs. (3.28) and (3.29), we get, $\frac{h}{k} \neq f(x)$ for fully developed flow (Fig. 3.7). Here k is assumed to be constant, hence h is independent of x.

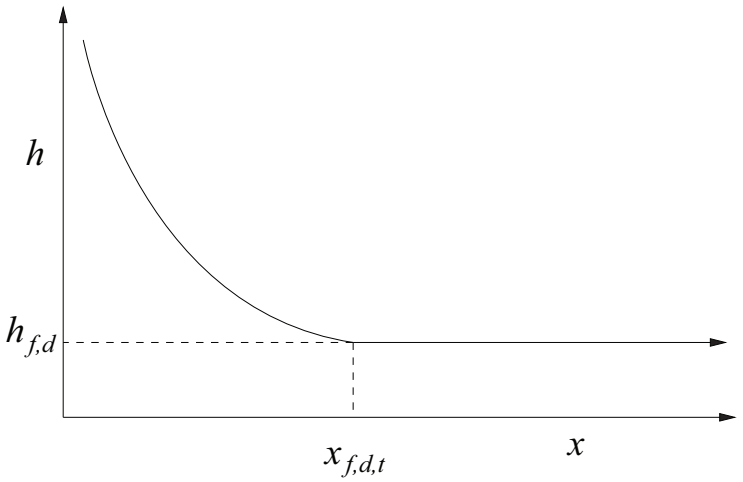

Figure 3.7 Heat transfer coefficient for thermally fully developed flow

Thus we conclude that the local heat transfer coefficient in a thermally fully developed region of pipe or tube does not vary with x.

3.3.3 Energy Balance in Ducted Flows

An energy balance may be applied to a pipe or tube to determine the variation of mean temperature (T_m) with the position along the tube. The total convective

energy transfer can be related to the difference in bulk-mean temperatures at the tube outlet and inlet. Figure 3.8 depicts energy balance.

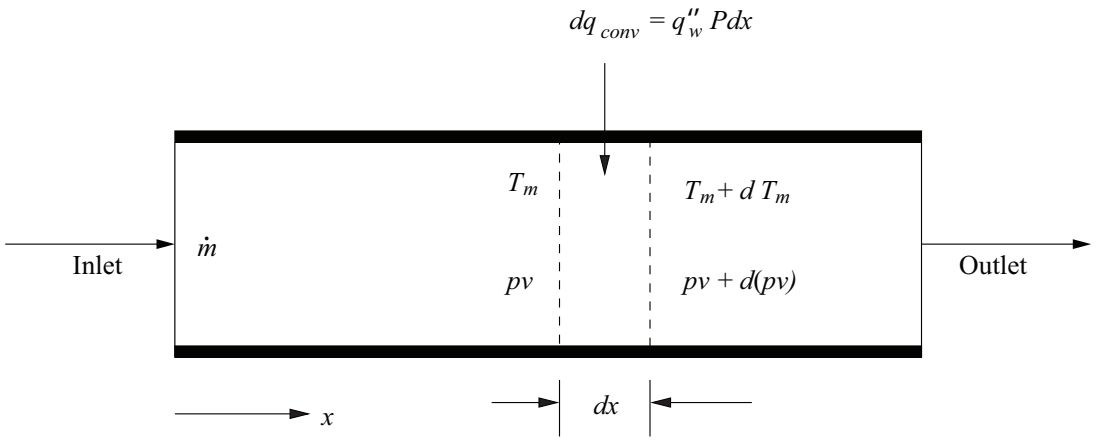

Figure 3.8 Energy balance for flow in a heated tube

$$dq_{conv} + \dot{m}(c_v T_m + pv) - \left[\dot{m}(c_v T_m + pv) + \dot{m}\frac{d(c_v T_m + pv)}{dx}dx\right] = 0$$

or

$$dq_{conv} = \dot{m}\; d(c_v T_m + pv) \tag{3.30}$$

We require finding the exact expression for dq for the compressible as well as incompressible flows. For compressible flows:

$$dq_{conv} = \dot{m}\; d(c_v T_m + pv)$$

or

$$dq_{conv} = \dot{m}\; d(c_v T_m + RT_m)$$

or

$$dq_{conv} = \dot{m}\; d[c_v T_m + (c_p - c_v)T_m]$$

or

$$dq_{conv} = \dot{m}\; c_p dT_m \tag{3.31}$$

For incompressible flows, $dq_{conv} = \dot{m}c_p dT_m$ [since $d(pv) = 0$ and $c_p = c_v$].

Therefore, irrespective of the condition the flow is compressible or incompressible, we can write

$$q_{conv} = \dot{m} c_p (T_{m,0} - T_{m,i}) \tag{3.32}$$

Between any two axial locations, we can write

$$dq_{conv} = \dot{m} c_p dT_m$$

or

$$q_w^{''} P dx = \dot{m} c_p dT_m \quad [\text{P is the surface perimeter} = \pi D]$$

or

$$\frac{dT_m}{dx} = \frac{q_w^{''} P}{\dot{m} c_p} = \frac{Ph(T_w - T_m)}{\dot{m} c_p} \tag{3.33}$$

When $T_w > T_m$, heat is transferred to fluid and T_m increases with x. The manner in which Eq. (3.33) varies is required to be understood.

Variation of T_m for uniform wall temperature:
The perimeter, P may vary with x, but it is constant for a pipe. $P/\dot{m} c_p$ is constant. For a fully developed flow, h is constant, although it varies with x in the entrance region. Finally T_w is constant. Therefore, T_m will vary with x except for the trivial case ($T_w = T_m$) of no heat transfer.

From Eq. (3.33), $\frac{dT_m}{dx} = q_w^{''} P/\dot{m} c_p = \frac{Ph(T_w - T_m)}{\dot{m} c_p}$, defining $\Delta T = T_w - T_m$ = difference between surface temperature and mean temperature.

$$\frac{dT_m}{dx} = -\frac{d(\Delta T)}{dx} = \frac{Ph(\Delta T)}{\dot{m} c_p} \tag{3.34}$$

or

$$\int_{\Delta T_i}^{\Delta T_0} \frac{d(\Delta T)}{(\Delta T)} = \frac{-P}{\dot{m} c_p} \int_0^L h dx$$

or

$$\ln \frac{(\Delta T_0)}{(\Delta T_i)} = -\frac{PL}{\dot{m} c_p} \cdot \frac{1}{L} \int_0^L h dx$$

or

$$\ln \frac{\Delta T_0}{\Delta T_i} = -\frac{PL}{\dot{m} c_p} \overline{h}_L$$

$$\frac{\Delta T_0}{\Delta T_i} = \frac{T_w - T_{m,0}}{T_w - T_{m,i}} = exp\left(-\frac{PL}{\dot{m} c_p} \bar{h}_L\right) \tag{3.35}$$

If we integrate from the tube inlet to some axial position x within the tube, we shall obtain a more general expression.

$$\frac{T_w - T_m(x)}{T_w - T_{m,i}} = \frac{T_w - T_m(x)}{T_w - T_{m,i}} = exp\left(-\frac{Px}{\dot{m}c_p}\bar{h}\right) \tag{3.36}$$

where $\bar{h}$ is the average value of h from the inlet to the distance x. This result suggests that the temperature difference $(T_w - T_m)$ decays exponentially (also clear from Fig. 3.9) in the flow direction.

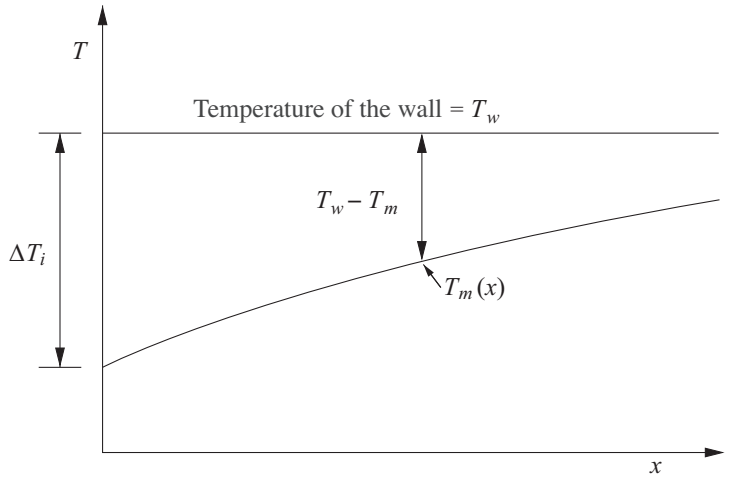

Figure 3.9 Variation of bulk-mean temperature of the fluid in a tube with uniform wall temperature boundary condition

Variation of T_m for uniform wall heat flux:

For uniform wall heat flux, $q_w^{''}$ is independent of x. Total heat transfer rate is $q_{conv}^{total} = q_w^{''}PL$. From Eqs. (3.30) through (3.32), $q_w^{''}PL = \dot{m}c_p(T_{m,0} - T_{m,i})$. Therefore if $q_w^{''}$, $\dot{m}$, c_p and the geometry is known, it is possible to calculate the fluid temperature rise $(T_{m,0} - T_{m,i})$ or we can write

$$\frac{dT_m}{dx} = \frac{q_w^{''}P}{\dot{m}c_p} \quad \text{or} \quad T_m = \frac{q_w^{''}P}{\dot{m}c_p}x + C_1$$

at $x = 0$, $T_m = T_{m,i}$; therefore $C_1 = T_{m,i}$.

$$T_m(x) = T_{m,i} + \frac{q_w^{''}P}{\dot{m}c_p}x \tag{3.37}$$

We can see the variation $T_m(x)$ with x for constant wall heat flux in Fig. 3.10.

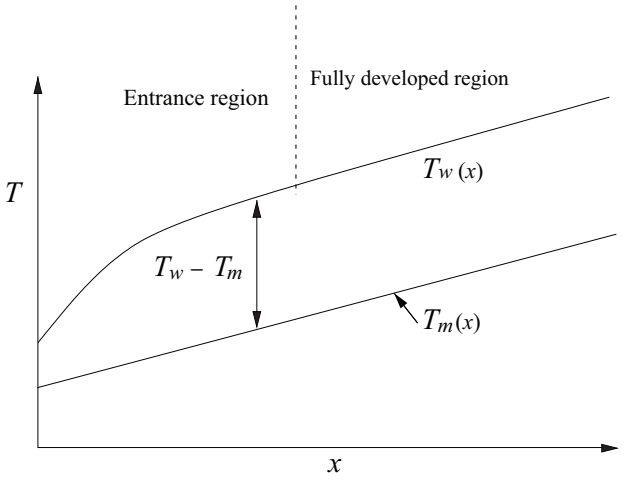

Figure 3.10 Variation of bulk-mean temperature of the fluid in a tube with uniform wall heat flux boundary condition

3.4 LAMINAR FLOW IN CIRCULAR TUBE

The problem of laminar flow in a circular tube has been treated theoretically and the results have been derived to express convection coefficients. We wish to calculate heat transfer under fully developed hydrodynamic and thermal conditions. At any point in the tube the boundary layer approximations may be applied.

3.4.1 Heat Transfer through Circular Tube for Hydrodynamically and Thermally Developed Flow with Uniform Wall Heat Flux (UHF) Condition

The governing energy equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{\alpha}{r}\ \frac{\partial}{\partial r}\ \left(r\frac{\partial T}{\partial r}\right) \tag{3.38}$$

Solutions of hydrodynamically fully developed flow are $v = 0, \partial u/\partial x = 0$ and $u(r) = 2u_m[1 - (r/r_o)^2]$. Substituting u and v in Eq. (3.38), and applying the thermal condition, we get

$$2u_m\ \left[1 - \left(\frac{r}{r_0}\right)^2\right]\frac{dT_m}{dx} = \frac{\alpha}{r}\ \frac{\partial}{\partial r}\ \left(r\frac{\partial T}{\partial r}\right) \tag{3.39}$$

Note that in the above equation, at any x, $\frac{dT}{dx} = \frac{dT_m}{dx}$. We are solving for T (at any x) with respect to r. At a given x (where dT_m/dx is known), T becomes a function of r only. We can write

$$\frac{1}{r}\ \frac{d}{dr}\ \left[r\frac{dT}{dr}\right] = \frac{2u_m}{\alpha}\ \frac{dT_m}{dx}\ \left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

or

$$r\frac{dT}{dr} = \frac{2u_m}{\alpha}\ \frac{dT_m}{dx}\left[\frac{r^2}{2} - \frac{r^4}{4r_0^2}\right] + C_1$$

or

$$\frac{dT}{dr} = \frac{2u_m}{\alpha}\ \frac{dT_m}{dx}\left[\frac{r}{2} - \frac{r^3}{4r_0^2}\right] + \frac{C_1}{r}$$

or

$$T(r) = \frac{2u_m}{\alpha}\ \frac{dT_m}{dx}\left[\frac{r^2}{4} - \frac{r^4}{16r_0^2}\right] + C_1\ \ln\ r + C_2 \tag{3.40}$$

At $r = 0$, T is finite; $C_1 = 0$. At $r = r_0,\ T = T_w$

$$C_2 = T_w - \frac{2u_m}{\alpha}\left(\frac{dT_m}{dx}\right)\ \frac{3r_0^2}{16}$$

$$T(r) = T_w - \frac{2u_m\ r_0^2}{\alpha}\ \left(\frac{dT_m}{dx}\right)\left[\frac{3}{16} + \frac{1}{16}\left(\frac{r}{r_0}\right)^4 - \frac{1}{4}\left(\frac{r}{r_0}\right)^2\right] \tag{3.41}$$

Now,

$$T_m = \frac{\int_0^{r_0}\ u(r)\ T(r)\ rdr}{\int_0^{r_0}\ u(r)\ \ rdr}$$

Substituting $(dT_m/dx) = (q_w^{''}P)/\dot{m}c_p$ in Eq. (3.41), we get from the above equation

$$T_m = \frac{\int_0^{r_0}\ 2u_m\left(1 - \frac{r^2}{r_0^2}\right) r\left[T_w + \frac{q_w^{''}r_0}{k}\left\{\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{3}{4}\right\}\right] dr}{\int_0^{r_0} 2u_m\left[1 - \left(\frac{r}{r_0}\right)^2\right] rdr}$$

$$T_m = \frac{2u_m\left[T_w\left(\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2}\right) + \frac{q_w^{''}r_0}{k}\int_0^{r_0}\left(r - \frac{r^3}{r_0^2}\right)\left(\frac{r^2}{r_0^2} - \frac{1}{4}\frac{r^4}{r_0^4} - \frac{3}{4}\right)\right] dr}{2u_m\left[\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2}\right]}$$

$$T_m - T_w = \frac{q_w^{''}r_0}{k}\left(-\frac{11}{24}\right) \tag{3.42}$$

$$\frac{q_w^{''}}{(T_w - T_m)}\frac{2r_0}{k_f} = Nu_D = \frac{2 \times 24}{11} = 4.36 \tag{3.43}$$

where $q_w^{''} = h(T_w - T_m) = k\frac{dT}{dr}\ |_{r=r_0}$.

For the UHF boundary condition, the fully developed laminar flow in a tube is $Nu_D = 4.36$ independent of Re and Pr. Finally, the fully developed temperature profile in a tube may be expressed as

$$T(r) = T_w + \frac{q_w'' r_0}{\kappa}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{3}{4}\right] \tag{3.44}$$

$$T(r) = T_m + \frac{q_w'' r_0}{\kappa}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24}\right] \tag{3.45}$$

3.4.2 Heat Transfer through Circular Tube for Thermally Fully Developed Laminar Slug Flow with Uniform Wall Temperature (UWT) Condition

There may be a flow situation when unheated entrance length is not short and the thermal diffusivity is more than the momentum diffusivity ($\nu/\ \propto << 1$); temperature profiles develop more rapidly than the velocity profile. In such a sitution, it is appropriate to assume axial velocity to be uniform across the cross-section. However, results developed under such assumption cannot be extended too far downstream.

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3.46}$$

Slug flow means u is independent of $r, u = U =$ constant

$$U\frac{\partial T}{\partial x} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \tag{3.47}$$

or

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{U}{\alpha}\frac{\partial T}{\partial x} \tag{3.48}$$

T is a function of x and r.

Substituting $\frac{T-T_w}{T_\infty - T_w} = \theta$ and $\frac{r}{r_0} = Y$ and $x = \frac{Ur_0^2}{\alpha}X$, we get

$$\frac{\partial^2 \theta}{\partial Y^2} + \frac{1}{Y}\frac{\partial \theta}{\partial Y} = \frac{\partial \theta}{\partial X} \tag{3.49}$$

Assume $\theta = F(X)G(Y)$ where $F = F(X)$ and $G = G(Y)$

$$F\frac{d^2 G}{dY^2} + \frac{F}{Y}\frac{dG}{dY} = G\frac{dF}{dX}$$

or

$$\frac{1}{G}\left[\frac{d^2 G}{dY^2} + \frac{1}{Y}\frac{dG}{dY}\right] = \frac{1}{F}\frac{dF}{dX} = -\beta_n^2$$